On the rate of convergence of ergodic averages for functions of Gordin class

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Koopman, von Neumann

$(\Omega, \mathfrak{F}, \mu)$ — standard probability measure space.
$T : \Omega \to \Omega$ — measurable invertible μ-preserving transformation (automorphism), i.e., $\forall A \in \mathfrak{F}$

$$T^{-1}A, TA \in \mathfrak{F} \quad \text{and} \quad \mu(A) = \mu(T^{-1}A) = \mu(TA).$$

$U_T f = f \circ T$ for $f \in L_2(\Omega, \mathfrak{F}, \mu)$ — Koopman operator.

$A_n f = \frac{1}{n} \sum_{k=1}^{n-1} U_T^k f$ — ergodic averages.

Theorem (von Neumann, 1932). For all $f \in L_2(\Omega, \mathfrak{F}, \mu)$

$$\|A_n f - \mathbb{E}(f|\mathfrak{I})\|_2 \to 0 \text{ as } n \to \infty.$$

Coboundaries

$f \in (I - U_T)L_2(\Omega, \mathfrak{F}, \mu)$ — coboundary, i.e., $f(x) = g(x) - g(Tx)$ for $g \in L_2(\Omega, \mathfrak{F}, \mu)$.

Coboundaries are U_T-invariant and dense in $L_2(\Omega, \mathfrak{F}, \mu) \ominus \mathbb{C}$.

Theorem (Browder, 1958; Butzer, Westphal, 1971).

$f - \mathbb{E}(f|\mathfrak{J})$ is coboundary iff $\|A_n f - \mathbb{E}(f|\mathfrak{J})\|_2 = \mathcal{O}(\frac{1}{n})$.

The rate of convergence $\mathcal{O}(\frac{1}{n})$ as $n \to \infty$ is maximal possible in the von Neumann ergodic theorem. Namely, the rate of convergence $o(\frac{1}{n})$ occurs only in the degenerate case $f = \mathbb{E}(f|\mathfrak{J})$ (Butzer, Westphal, 1971; Gaposhkin, 1975; Sedalishchev, 2014)

Filtration

Let $\mathfrak{F}_0$ be a σ-subalgebra of σ-algebra $\mathfrak{F}$ such that $T^{-1}\mathfrak{F}_0 \subseteq \mathfrak{F}_0$. Thus, we have filtration of σ-algebras $\mathfrak{F}_n := T^n\mathfrak{F}_0, n \in \mathbb{Z}$, i.e.,

$$\mathfrak{F}_n \subseteq \mathfrak{F}_{n+1}, \ n \in \mathbb{Z}.$$

Let

$$\mathfrak{F}_{-\infty} = \bigcap_{n<0} \mathcal{F}_n, \quad \mathfrak{F}_{+\infty} = \bigvee_{n\geq 0} \mathfrak{F}_n,$$

and denote $E_n f = \mathbb{E}(f|\mathfrak{F}_n)$, $n \in [-\infty, +\infty]$.

Theorem (Adler, 1964; Sinai, 1964). The following equivalence is valid: for any σ-algebra $\mathfrak{F}_0 \subseteq \mathfrak{F}$

$$[T^{-1}\mathfrak{F}_0 \subseteq \mathfrak{F}_0 \quad \Rightarrow \quad \mathfrak{F}_0 = T^{-1}\mathfrak{F}_0] \quad \Leftrightarrow \quad h(T) = 0.$$

Here $h(T)$ is measure-theoretic entropy of T.

Gordin classes and Gordin space

We call a function f belonging to *the Gordin class* $\mathfrak{G}(T, \mathfrak{F}_0)$ generated by the σ-algebra $\mathfrak{F}_0$, if

$$f \in (I - E_n) L_2(\Omega, \mathfrak{F}_m, \mu)$$

for some $m, n \in \mathbb{Z}, m > n$. Thus,

$$\mathfrak{G}(T, \mathfrak{F}_0) = \bigcup_{m>n} H_{n,m}, \quad H_{n,m} = (I - E_n) L_2(\Omega, \mathfrak{F}_m, \mu).$$

The Gordin space $\mathfrak{G}(T)$ is the linear span of all Gordin classes $\mathfrak{G}(T, \mathfrak{F}_0)$, i.e.,

$$\mathfrak{G}(T) = \mathrm{span}\{\mathfrak{G}(T, \mathfrak{F}_0) : \ T^{-1}\mathfrak{F}_0 \subseteq \mathfrak{F}_0\}.$$

Why such a name?

Due to famous Gordin paper in CLT, where this constrution works.
M. I. Gordin, "The central limit theorem for stationary processes", Dokl. Akad. Nauk SSSR, 188:4 (1969), 739–741.

• The proof of Kolmogorov's theorem that K-automorphisms have infinite Lebesgue spectral type involves functions from the Gordin class of the form $\chi_A - \mathbb{E}(\chi_A|\mathfrak{F}_{-1}), A \in \mathfrak{F}_0$.

• Assani showed that functions of the form $g = \chi_A - \mathbb{E}(\chi_A|\mathfrak{F}_{-n})$, $A \in \mathfrak{F}_m, m, n > 0$ are Wiener-Wintner functions of power type 1/4 in $L_2(\Omega, \mathfrak{F}, \mu)$. This means that

$$\left\| \sup_{\varepsilon} \left| \frac{1}{N} \sum_{k=0}^{N-1} g \circ T^k \cdot e^{2\pi i k \varepsilon} \right| \right\|_2 = \mathcal{O}\left(\frac{1}{\sqrt[4]{N}} \right).$$

The rate of convergence

Theorem (P., 2023). Let the σ-algebra $\mathfrak{F}_0$ be such that $\mathfrak{J} \subset \mathfrak{F}_n$ for each $n \in \mathbb{Z}$. Then for any function f from the Gordin class $\mathfrak{G}(T, \mathfrak{F}_0)$ we have

$$\|A_N f\|_2 = \mathcal{O}\left(\frac{1}{\sqrt{N}}\right) \quad \text{as} \quad N \to \infty.$$

Sketch: There are $n, m \in \mathbb{Z}$ with $m > n$ and $g \in L_2(\Omega, \mathfrak{F}_m, \mu)$ such that $f = g - E_n g$. Then $\mathbb{E}(f|\mathfrak{J}) = 0$.

We have $\|A_N f\|_2^2 = \dfrac{\|f\|_2^2}{N} + \dfrac{2}{N^2} \displaystyle\sum_{k=1}^{N-1} (N-k) \mathrm{Re}(f \circ T^k, f)_2,$ but

for $k \geq m - n$ the function $g \circ T^k$ will be $\mathfrak{F}_{m-k}$-measurable. It yelds $(f \circ T^k, f)_2 = 0$ for $k \geq m - n$.

Corollaries

Corollary 1. For any function f from the Gordin space, the spectral measure σ_f satisfies the estimate $\sigma_f((-\delta,\delta]) = \mathcal{O}(\delta)$ as $\delta \to 0$ (using Kachurovskii criterion, 1996).

Corollary 2. For uniform convergence on the space $H_{n,m}$ there is the estimate

$$\|A_N\|_{H_{n,m}\to L_2(\Omega,\mathfrak{F},\mu)} \leq \sqrt{\frac{m-n}{N}} \quad \text{for all} \quad N \geq m-n.$$

Corollary 3. For any function f from the Gordin space, the asymptotic relation is valid a.e.

$$A_N f = o\left(\frac{\ln N(\ln\ln N)^{\beta}}{\sqrt{N}}\right) \quad \text{as} \quad N \to \infty$$

for any $\beta > 1/2$ (using Cuny theorem, 2011).

Properties of the Gordin classes

Let $h(T) > 0$, then the Gordin class $\mathfrak{G}(T, \mathfrak{F}_0)$:

- is a linear space;
- is invariant under the Koopman operator;
- is a dense subset of the first Baire category in

$$L_2(\Omega, \mathfrak{F}_{+\infty}, \mu) \ominus L_2(\Omega, \mathfrak{F}_{-\infty}, \mu).$$

It follows from:

$$H_{m,n} + H_{p,q} \subset H_{\max\{m,p\},\min\{n,q\}}, \qquad U_T H_{n,m} = H_{n-1,m-1},$$

and using Doob convergence theorem.

Pinsker algebra

The condition $h(T) = 0$ can also be expressed by the following equality of σ-algebras:

$$\Pi(T, \mathfrak{F}) = \mathfrak{F},$$

where $\Pi(T, \mathfrak{F})$ is the Pinsker σ algebra, i.e.,

$$\Pi(T, \mathfrak{F}) = \{A \in \mathfrak{F} : \ h(\{A, \Omega \setminus A\} = 0)\}.$$

Thus, for automorphisms with $h(T) > 0$, the Pinsker σ-algebra $\Pi(T, \mathfrak{F})$ is a proper σ-subalgebra of the σ-algebra $\mathfrak{F}$.

Theorem (Rokhlin, Sinai, 1961). Let $\mathfrak{F}'$ be a T-invariant σ-algebra, i.e., $T^{-1}\mathfrak{F}' = \mathfrak{F}'$. Then there exists an extremal σ-subalgebra $\mathfrak{F}_0 \subset \mathfrak{F}'$ such that

$$T^{-1}\mathfrak{F}_0 \subset \mathfrak{F}_0, \quad \mathfrak{F}_{-\infty} = \Pi(T, \mathfrak{F}'), \quad \mathfrak{F}_{+\infty} = \mathfrak{F}'.$$

On the closure of Gordin space

Let $h(T) > 0$ and T be ergodic, then

$$
\begin{aligned}
c\ell\,\mathfrak{G}(T) &= c\ell\,\mathsf{span}\{\mathfrak{G}(T,\mathfrak{F}_0) : \; T^{-1}\mathfrak{F}_0 \subseteq \mathfrak{F}_0\} \\
&= c\ell\,\mathsf{span}\{L_2(\Omega,\mathfrak{F}_{+\infty},\mu) \ominus L_2(\Omega,\mathfrak{F}_{-\infty},\mu) : \; T^{-1}\mathfrak{F}_0 \subseteq \mathfrak{F}_0\} \\
&= c\ell\,\mathsf{span}\{L_2(\Omega,\mathfrak{F}',\mu) \ominus L_2(\Omega,\Pi(T,\mathfrak{F}'),\mu) : \; T^{-1}\mathfrak{F}' = \mathfrak{F}'\}
\end{aligned}
$$